Applied Mathematics Letters 22 (2009) 428–434

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

On generalized variational-like inequalities with generalized monotone multivalued mappings

L.C. Ceng [a], J.C. Yao [b,*]

[a] Department of Mathematics, Shanghai Normal University, Shanghai, China

[b] Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung, Taiwan

ARTICLE INFO

Article history:
Received 7 September 2006
Received in revised form 30 March 2008
Accepted 3 June 2008

Keywords:
Generalized variational-like inequality
Generalized monotone multivalued mapping
KKM technique
Hausdorff metric
Reflexive Banach space

ABSTRACT

Let E be a reflexive Banach space with the dual space E^* and K be a nonempty closed convex subset of E. Let us have $\Psi : K \times K \times E^* \to R$ and $A : E^* \to E^*$. We introduce the class of generalized α-monotone multifunctions $T : K \to 2^{E^*}$ with respect to Ψ and A where $\alpha : E \times E \to R$. By using the KKM technique and the concept of the Hausdorff metric, we establish some existence results for generalized variational-like inequalities with generalized monotone multivalued mappings in E.

1. Introduction

It is well known that the KKM technique has played a very important role in the study of many fields such as optimization and mathematical programming problems, equilibrium problems, game theory, variational inequality theory and so on; see Refs. [1–9]. Let X be a Hausdorff topological vector space, and let D be a nonempty subset of X. A multivalued mapping $T : D \to 2^X$ is called a KKM map if $\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subset \cup_{i=1}^n T(x_i)$ for each finite subset $\{x_1, x_2, \ldots, x_n\} \subset D$, where $\mathrm{co}\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of the set $\{x_1, x_2, \ldots, x_n\}$. In Ref. [10], Fan proved the following celebrated lemma which asserts that, given an arbitrary set D in X and a KKM mapping $T : D \to 2^X$, if T has closed values and Tx is compact for at least one $x \in D$, then $\cap_{x\in D} Tx \neq \emptyset$.

In 1997, by using the KKM technique Konnov and Yao proved in Ref. [4] some results about the existence of solutions for vector variational inequalities with C_x-pseudomonotone multivalued mappings which were extended later by Ansari, Siddiqi and Yao in Ref. [1]. In 1999, Chen (Ref. [11]) obtained the existence of solutions for a class of variational inequalities with semi-monotone single-valued maps in nonreflexive Banach spaces. In 2003, Fang and Huang (Ref. [2]) considered two classes of variational-like inequalities with generalized monotone and semi-monotone mappings. Utilizing the KKM technique, they proved the existence of solutions for these variational-like inequalities with relaxed η–α-monotone mappings in reflexive Banach spaces.

In this work, let K be a nonempty closed convex subset of a reflexive Banach space E. Let us have $T : K \to 2^{E^*}$, $A : E^* \to E^*, f : K \to R$ and $\alpha : E \times E \to R$. We consider the following problems (I) and (II):

(I) Find $x \in K$ such that for each $y \in K$, there exists $s \in Tx$ satisfying

$$\Psi(y, x; As) + f(y) - f(x) \geq 0.$$

* Corresponding author.
E-mail address: yaojc@math.nsysu.edu.tw (J.C. Yao).

0893-9659/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2008.06.011

(II) Find $x \in K$ such that

$$\Psi(y, x; At) + f(y) - f(x) \geq \alpha(x, y), \quad \forall y \in K, \forall t \in Tx.$$

In particular, if we put $\Psi(x, y; z^*) = \langle z^*, \eta(x, y)\rangle$ for all $(x, y, z^*) \in K \times K \times E^*$ where $\eta : K \times K \to E$, then problems (I) and (II) reduce to problems (III) and (IV), respectively:

(III) Find $x \in K$ such that for each $y \in K$, there exists $s \in Tx$ satisfying

$$\langle As, \eta(y, x)\rangle + f(y) - f(x) \geq 0.$$

(IV) Find $x \in K$ such that

$$\langle At, \eta(y, x)\rangle + f(y) - f(x) \geq \alpha(x, y), \quad \forall y \in K, \forall t \in Tx.$$

If T is single-valued, then problems (III) and (IV) reduce to the problems studied in Ref. [2]. We will introduce the class of generalized α-monotone multifunctions $T : K \to 2^{E^*}$ with respect to Ψ and A. By using the KKM technique and the concept of the Hausdorff metric, we establish some existence results for generalized variational-like inequalities with generalized monotone multivalued mappings in E.

2. Preliminaries

Throughout this work, we consider the real reflexive Banach space E and its dual space E^*. Let K be a nonempty closed convex subset of E. We consider the mappings $A : E^* \to E^*$ and $\eta : K \times K \to E$, the multivalued mapping $T : K \to 2^{E^*}$ and the functions $\Psi : K \times K \times E^* \to R$, $\alpha : E \times E \to R$ and $f : K \to R$.

Definition 1. T is called generalized α-monotone with respect to Ψ and A if for any $x, y \in K$ we have

$$\Psi(y, x; At) - \Psi(y, x; As) \geq \alpha(x, y) \tag{1}$$

for each $s \in Tx$ and $t \in Ty$, where $\lim_{t\to 0^+} \frac{\alpha(x,x+t(y-x))}{t} = 0$.

Definition 2. Let $\Psi(x, y; z^*) = \langle z^*, \eta(x, y)\rangle$ for each $(x, y, z^*) \in K \times K \times E^*$. The multivalued mapping T is called generalized η–α-monotone with respect to A if the inequality (1) holds.

Remark 1. (i) If $\Psi(x, y; z^*) = \langle z^*, \eta(x, y)\rangle$, $A = I$ the identity mapping of E^* and T is single-valued, then Definition 1 reduces to general η–α monotonicity (see Ref. [3]).

(ii) In the case of (i), if $\alpha(x, y) = \beta(y - x)$, where $\beta : K \to R$ with $\beta(\lambda z) = \lambda^p \beta(z)$ for $\lambda > 0$, $p > 1$, then the case (i) reduces to relaxed η–α monotonicity of mapping T (see, e.g., Ref. [2]).

(iii) In the case of (ii), if $\eta(x, y) = x - y$ for each $(x, y) \in K \times K$, then the case (ii) reduces to

$$\langle Ty - Tx, y - x\rangle \geq \beta(y - x), \quad \forall x, y \in K,$$

and T is called relaxed α-monotone (see, e.g., Ref. [2]).

(iv) In the case of (iii), if $\beta(z) = k\|z\|^p$, where $k > 0$ is a constant, then the case (iii) reduces to

$$\langle Ty - Tx, y - x\rangle \geq k\|x - y\|^p, \quad \forall x, y \in K,$$

and T is called p-monotone (see, e.g., Ref. [4]).

Definition 3. Ψ is f-coercive with respect to T and A if there exists $y_0 \in K$ such that

$$\lim_{\|x\|\to\infty} \inf_{s\in Tx} \frac{\Psi(x, y_0; As) - \Psi(x, y_0; At_0) + f(x) - f(y_0)}{|\Psi(y_0, x; At_0)|} = +\infty$$

for some $t_0 \in Ty_0$.

Remark 2. (i) If $f \equiv 0$, $A = I$ and T is single-valued, then $\Psi(\cdot, \cdot; T(\cdot)) : K \times K \times K \to R$ is called coercive (see Ref. [3]), i.e., there exists $y_0 \in K$ such that

$$\lim_{\|x\|\to\infty} \frac{\Psi(x, y_0; Tx) - \Psi(x, y_0; Ty_0)}{|\Psi(y_0, x; Ty_0)|} = +\infty.$$

(ii) If $\Psi(x, y; z^*) = \langle z^*, \eta(x, y)\rangle$, $f \equiv 0$, $A = I$ and T is single-valued, then the condition in Definition 3 reduces to

$$\lim_{\|x\|\to\infty} \frac{\langle Tx, \eta(x, y_0)\rangle - \langle Ty_0, \eta(x, y_0)\rangle}{|\langle Ty_0, \eta(y_0, x)\rangle|} = +\infty.$$

(iii) If $\Psi(x, y; z^*) = \langle z^*, \eta(x, y)\rangle$, $A = I$ and T is single-valued, then Definition 3 reduces to η-coercivity of the mapping T with respect to f under suitable condition (see Ref. [2]). In this case, if $f = \delta_K$ where δ_K is the indicator function of K, then Definition 3 coincides with the definition of η-coercivity in the sense of Konnov and Yao (see Ref. [4] and also Ref. [5]).

Lemma 1 *(See Ref. [12]). Let $(E, \|\cdot\|)$ be a normed vector space and H be a Hausdorff metric on the collection $CB(E)$ of all closed and bounded subsets of E, induced by a metric d in terms of $d(x, y) = \|x - y\|$ which is defined by*

$$H(A, B) = \max(\sup_{x\in A}\inf_{y\in B}\|x - y\|, \sup_{y\in B}\inf_{x\in A}\|x - y\|),$$

for A and B in $CB(E)$. If A and B are any two members in $CB(E)$, then for each $\varepsilon > 0$ and each $x \in A$, there exists $y \in B$ such that

$$\|x - y\| \le (1 + \varepsilon)H(A, B).$$

In particular, if A and B are any two compact subsets in E, then for each $x \in A$, there exists $y \in B$ such that

$$\|x - y\| \le H(A, B).$$

Lemma 2. *Fan's lemma (Ref. [10]). Let D be an arbitrary set in a Hausdorff topological vector space X. Let $T : D \to 2^X$ be a KKM map such that Tx is closed for all $x \in D$ and is compact for at least one $x \in D$. Then $\bigcap_{x\in D} Tx \neq \emptyset$.*

3. Main results

First, we have the following type of Minty's lemma for problems (I) and (II).

Theorem 1. *Let $T : K \to 2^{E^*}$ be a nonempty compact-valued multifunction such that for any $x, y \in K$,*

$$H(T(x + \lambda(y - x)), Tx) \to 0 \quad \text{as } \lambda \to 0^+,$$

where H is a Hausdorff metric defined on $CB(E^*)$. Suppose that the following conditions hold:

(i) *$A : E^* \to E^*$ is a continuous mapping;*
(ii) *$f : K \to R$ is a lower semicontinuous and convex functional;*
(iii) *$\Psi(x, \cdot; \cdot) : K \times E^* \to R$ is continuous for each fixed $x \in K$;*
(iv) *$\Psi(x, y; z^*) + \Psi(y, x; z^*) = 0$ for each $(x, y, z^*) \in K \times K \times E^*$;*
(v) *$\Psi(\cdot, y; At)$ is a convex function on K for each $y \in K$ and $t \in Ty$;*
(vi) *T is generalized α-monotone with respect to Ψ and A.*

Then problems (I) and (II) are equivalent.

Proof. Suppose that problem (I) has a solution; i.e., there exists an $x_0 \in K$ such that for any $y \in K$ there is $s_0 \in Tx_0$ satisfying

$$\Psi(y, x_0; As_0) + f(y) - f(x_0) \ge 0.$$

Since T is generalized α-monotone with respect to Ψ and A, it follows that for all $y \in K$ and $t \in Ty$

$$\begin{aligned}\Psi(y, x_0; At) + f(y) - f(x_0) &\ge \Psi(y, x_0; As_0) + \alpha(x_0, y) + f(y) - f(x_0)\\ &\ge \alpha(x_0, y).\end{aligned}$$

This shows that problem (II) has a solution.

Conversely, suppose that problem (II) has a solution; i.e., there exists an $x_0 \in K$ such that

$$\Psi(y, x_0; At) + f(y) - f(x_0) \ge \alpha(x_0, y)$$

for all $y \in K$ and $t \in Ty$. For an arbitrary $y \in K$, letting $y_\lambda = \lambda y + (1 - \lambda)x_0$, $0 < \lambda < 1$, we have $y_\lambda \in K$ by the convexity of K. Hence for all $t_\lambda \in Ty_\lambda$,

$$\Psi(y_\lambda, x_0; At_\lambda) + f(y_\lambda) - f(x_0) \ge \alpha(x_0, y_\lambda). \tag{2}$$

According to conditions (ii), (iv), (v), we have

$$\begin{aligned}0 &= \Psi(y_\lambda, y_\lambda; At_\lambda) + f(y_\lambda) - f(y_\lambda)\\ &= \Psi(\lambda y + (1 - \lambda)x_0, y_\lambda; At_\lambda) + f(\lambda y + (1 - \lambda)x_0) - f(y_\lambda)\\ &\le \lambda\Psi(y, y_\lambda; At_\lambda) + \lambda f(y) - \lambda f(y_\lambda) + (1 - \lambda)\Psi(x_0, y_\lambda; At_\lambda) + (1 - \lambda)f(x_0) - (1 - \lambda)f(y_\lambda),\end{aligned}$$

which implies that

$$\begin{aligned}\Psi(y, y_\lambda; At_\lambda) + f(y) - f(y_\lambda) &\geq \frac{1-\lambda}{\lambda}[-\Psi(x_0, y_\lambda; At_\lambda) + f(y_\lambda) - f(x_0)] \\ &= \frac{1-\lambda}{\lambda}[\Psi(y_\lambda, x_0; At_\lambda) + f(y_\lambda) - f(x_0)] \\ &\geq \frac{\alpha(x_0, y_\lambda)}{\lambda}(1-\lambda). \end{aligned} \tag{3}$$

Since Ty_λ, Tx_0 are compact, by Lemma 1 for each $t_\lambda \in Ty_\lambda$ we can find an $s_\lambda \in Tx_0$ such that

$$\|t_\lambda - s_\lambda\| \leq H(Ty_\lambda, Tx_0).$$

Since Tx_0 is compact, without loss of generality, we may assume that $s_\lambda \to s_0 \in Tx_0$ as $\lambda \to 0^+$. Moreover, we have

$$\begin{aligned}\|t_\lambda - s_0\| &\leq \|t_\lambda - s_\lambda\| + \|s_\lambda - s_0\| \\ &\leq H(Ty_\lambda, Tx_0) + \|s_\lambda - s_0\|.\end{aligned}$$

Since $H(Ty_\lambda, Tx_0) \to 0$ as $\lambda \to 0^+$, so $t_\lambda \to s_0$. Since f is lower semicontinuous, we have

$$\liminf_{\lambda\to 0^+} f(y_\lambda) \geq f(x_0).$$

Also, since $At_\lambda \to As_0$ and $y_\lambda \to x_0$ as $\lambda \to 0^+$, we have $\Psi(y, y_\lambda; At_\lambda) \to \Psi(y, x_0; As_0)$ as $\lambda \to 0^+$. Note that $\lim_{\lambda\to 0^+} \alpha(x_0, x_0 + \lambda(y - x_0))/\lambda = 0$. Thus from (3) we derive

$$\begin{aligned}\Psi(y, x_0; As_0) + f(y) - f(x_0) &\geq \limsup_{\lambda\to 0^+} \Psi(y, y_\lambda; At_\lambda) + f(y) - \liminf_{\lambda\to 0^+} f(y_\lambda) \\ &\geq \limsup_{\lambda\to 0^+} [\Psi(y, y_\lambda; At_\lambda) + f(y) - f(y_\lambda)] \\ &\geq \limsup_{\lambda\to 0^+} \frac{\alpha(x_0, y_\lambda)}{\lambda}(1-\lambda) \\ &= 0.\end{aligned}$$

This implies that problem (I) has a solution. □

If we put $\Psi(x, y; z^*) = \langle z^*, \eta(x, y)\rangle$ for all $(x, y, z^*) \in K \times K \times E^*$ where $\eta : K \times K \to E$, then the following corollary follows immediately from Theorem 1.

Corollary 1. *Let $T : K \to 2^{E^*}$ be a nonempty compact-valued multifunction such that for any $x, y \in K$,*

$$H(T(x + \lambda(y - x)), Tx) \to 0 \quad \text{as } \lambda \to 0^+,$$

where H is a Hausdorff metric defined on $CB(E^*)$. Suppose that the following conditions hold:

(i) *$A : E^* \to E^*$ is a continuous mapping;*
(ii) *$f : K \to R$ is a lower semicontinuous and convex functional;*
(iii) *$\eta(x, \cdot) : K \to E$ is continuous for each fixed $x \in K$;*
(iv) *$\eta(x, y) + \eta(y, x) = 0$ for each $(x, y) \in K \times K$;*
(v) *$(\langle At, \eta(\cdot, y)\rangle) : K \to R$ is a convex function on K for each $y \in K$ and $t \in Ty$;*
(vi) *T is generalized η–α-monotone with respect to A.*

Then problems (III) *and* (IV) *are equivalent.*

We now state and prove the following existence result for problem (I) by employing Theorem 1.

Theorem 2. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E and let $T : K \to 2^{E^*}$ be a nonempty compact-valued multifunction such that for any $x, y \in K$,*

$$H(T(x + \lambda(y - x)), Tx) \to 0 \quad \text{as } \lambda \to 0^+,$$

where H is a Hausdorff metric defined on $CB(E^*)$. Suppose that the following conditions hold:

(i) *$A : E^* \to E^*$ is a continuous mapping;*
(ii) *$f : K \to R$ is a lower semicontinuous and convex functional;*
(iii) *$\Psi(x, \cdot; \cdot) : K \times E^* \to R$ is continuous for each fixed $x \in K$;*
(iv) *$\Psi(x, y; z^*) + \Psi(y, x; z^*) = 0$ for each $(x, y, z^*) \in K \times K \times E^*$;*
(v) *$\Psi(\cdot, y; At)$ is a convex and lower semicontinuous function on K for each fixed $y \in K$ and $t \in Ty$;*
(vi) *T is generalized α-monotone with respect to Ψ and A;*

(vii) *$\alpha(\cdot, y)$ is weakly lower semicontinuous for each fixed $y \in K$, i.e., for each sequence $\{x_\nu\}_\nu$ that converges to x in $\sigma(E, E^*)$ one has*

$$\alpha(x, y) \le \liminf_{\nu\to\infty} \alpha(x_\nu, y), \quad \forall y \in K.$$

Then problem (I) *has a solution.*

Proof. We define two set-valued mappings $F, G : K \to 2^K$ as follows:

$$F(y) := \{x \in K : \text{ there exists } s \in Tx \text{ such that } \Psi(y, x; As) + f(y) - f(x) \ge 0\}, \quad \forall y \in K,$$

and

$$G(y) := \{x \in K : \Psi(y, x; At) + f(y) - f(x) \ge \alpha(x, y), \forall t \in Ty\}, \quad \forall y \in K.$$

Observe that $F(y)$ is nonempty for each $y \in K$, since $y \in F(y)$. Moreover, according to Theorem 1 we have

$$\bigcap_{y\in K} F(y) = \bigcap_{y\in K} G(y). \tag{4}$$

Next, we shall prove that

$$\bigcap_{y\in K} G(y) \ne \emptyset. \tag{5}$$

We claim first that F is a KKM mapping on K.

Suppose that F is not a KKM mapping. Then there exist $\{y_1, y_2, \dots, y_n\} \subset K$ and $\lambda_i \ge 0$, $i = 1, 2, \dots, n$, with $\sum_{i=1}^n \lambda_i = 1$ such that $y_0 = \sum_{i=1}^n \lambda_i y_i \notin \cup_{i=1}^n F(y_i)$. Hence it follows that $y_0 \notin F(y_i)$ for all $i = 1, 2, \dots, n$, i.e.,

$$\Psi(y_i, y_0; As_0) + f(y_i) - f(y_0) < 0, \quad \forall s_0 \in Ty_0, \tag{6}$$

for each $i = 1, 2, \dots, n$. From condition (iv) we obtain

$$\Psi(y, y; z^*) = 0, \quad \forall (y, z^*) \in K \times E^*.$$

Now, from conditions (ii), (v) and (6) it follows that

$$\begin{aligned} 0 &= \Psi(y_0, y_0; As_0) + f(y_0) - f(y_0) \\ &= \Psi\left(\sum_{i=1}^n \lambda_i y_i, y_0; As_0\right) + f\left(\sum_{i=1}^n \lambda_i y_i\right) - f(y_0) \\ &\le \sum_{i=1}^n \lambda_i \Psi(y_i, y_0; As_0) + \sum_{i=1}^n \lambda_i f(y_i) - f(y_0) \\ &= \sum_{i=1}^n \lambda_i [\Psi(y_i, y_0; As_0) + f(y_i) - f(y_0)] \\ &< 0, \end{aligned}$$

which yields a contradiction. Thus F is a KKM mapping.

We prove now that G is also a KKM mapping. It is sufficient to prove that for all $y \in K$ we have

$$F(y) \subset G(y).$$

Let $y \in K$. For each $x \in F(y)$ there exists $s \in Tx$ such that

$$\Psi(y, x; As) + f(y) - f(x) \ge 0.$$

Since T is generalized α-monotone with respect to Ψ and A, we deduce that for all $t \in Ty$

$$\Psi(y, x; At) + f(y) - f(x) \ge \Psi(y, x; As) + \alpha(x, y) + f(y) - f(x) \ge \alpha(x, y)$$

which implies that $x \in G(y)$. Consequently, $F(y) \subset G(y)$ for all $y \in K$ and hence G is a KKM mapping.

We prove now that $G(y)$ is weakly compact in K for each $y \in K$. Indeed, according to the definition of $G(y)$ and by conditions (ii) and (v) we conclude that the mapping $x \to \Psi(x, y; At) + f(x) - f(y)$ is weakly lower semicontinuous for each $y \in K$ and $t \in Ty$. From condition (iv) we conclude that for all $y \in K$

$$\begin{aligned} G(y) &= \{x \in K : \Psi(y, x; At) + f(y) - f(x) \ge \alpha(x, y), \forall t \in Ty\} \\ &= \{x \in K : \Psi(x, y; At) + f(x) - f(y) + \alpha(x, y) \le 0, \forall t \in Ty\}. \end{aligned}$$

Utilizing the weakly lower semicontinuity of $\alpha(\cdot, y)$ for each $y \in K$, we infer that $G(y)$ is weakly closed for all $y \in K$. Since K is a bounded closed and convex subset of E, it follows from the reflexivity of E that K is weakly compact, and so $G(y)$ is weakly compact in K for all $y \in K$. Using Lemma 2, we have that (5) holds. Therefore, according to (4) and (5) we get

$$\bigcap_{y\in K} F(y) \neq \emptyset,$$

and so there exists $\hat{x} \in K$ such that for any $y \in K$ there is $\hat{s} \in T\hat{x}$ satisfying

$$\Psi(y, \hat{x}; A\hat{s}) + f(y) - f(\hat{x}) \geq 0.$$

Consequently $\hat{x}$ is a solution of problem (I) and the theorem is proved. □

We next consider the case of unbounded closed convex domains.

Theorem 3. *Let K be a nonempty closed convex subset of a real reflexive Banach space E and let $T : K \to 2^{E^*}$ be a nonempty compact-valued multifunction such that for any $x, y \in K$,*

$$H(T(x + \lambda(y - x)), Tx) \to 0 \quad \text{as } \lambda \to 0^+,$$

where H is a Hausdorff metric defined on $CB(E^*)$. Assume that conditions (i)–(vii) *of* Theorem 2 *are fulfilled together with:*

(viii) *Ψ is f-coercive with respect to T and A.*

Then problem (I) *has a solution.*

Proof. For a positive real number r, we define

$$B_r := \{y \in E : \|y\| \leq r\}$$

and we consider the following problem: Find $x_r \in K \cap B_r$ such that for any $y \in K \cap B_r$ there is $s_r \in Tx_r$ satisfying

$$\Psi(y, x_r; As_r) + f(y) - f(x_r) \geq 0. \tag{7}$$

According to Theorem 2 we have that problem (7) has a solution $x_r \in K \cap B_r$. We claim that there exists $r' > 0$ such that $\|x_{r'}\| < r'$. If $\|x_r\| = r$ for each $r > 0$, then we choose r_0 such that $r_0 > \|y_0\|$ where y_0 is given by the coercivity condition of Ψ with respect to T and A. In this case we have

$$\Psi(y_0, x_{r_0}; As_{r_0}) + f(y_0) - f(x_{r_0}) \geq 0, \tag{8}$$

for some $s_{r_0} \in Tx_{r_0}$.

On the other hand, by condition (iv) in Theorem 2, we deduce that

$$\begin{aligned}
&\Psi(y_0, x_{r_0}; As_{r_0}) + f(y_0) - f(x_{r_0}) = -\Psi(x_{r_0}, y_0; As_{r_0}) + f(y_0) - f(x_{r_0}) \\
&\quad = -[\Psi(x_{r_0}, y_0; As_{r_0}) - \Psi(x_{r_0}, y_0; At_0) + f(x_{r_0}) - f(y_0)] - \Psi(x_{r_0}, y_0; At_0) \\
&\quad = -[\Psi(x_{r_0}, y_0; As_{r_0}) - \Psi(x_{r_0}, y_0; At_0) + f(x_{r_0}) - f(y_0)] + \Psi(y_0, x_{r_0}; At_0) \\
&\quad \leq -[\Psi(x_{r_0}, y_0; As_{r_0}) - \Psi(x_{r_0}, y_0; At_0) + f(x_{r_0}) - f(y_0)] + |\Psi(y_0, x_{r_0}; At_0)| \\
&\quad = -|\Psi(y_0, x_{r_0}; At_0)| \cdot \left[\frac{\Psi(x_{r_0}, y_0; As_{r_0}) - \Psi(x_{r_0}, y_0; At_0) + f(x_{r_0}) - f(y_0)}{|\Psi(y_0, x_{r_0}; At_0)|} - 1\right] \\
&\quad \leq -|\Psi(y_0, x_{r_0}; At_0)| \cdot \left[\inf_{s\in Tx_{r_0}} \frac{\Psi(x_{r_0}, y_0; As) - \Psi(x_{r_0}, y_0; At_0) + f(x_{r_0}) - f(y_0)}{|\Psi(y_0, x_{r_0}; At_0)|} - 1\right].
\end{aligned}$$

Now, we can choose r large enough so that the last inequality and the f-coercivity of Ψ with respect to T and A imply that

$$\Psi(y_0, x_{r_0}; As_{r_0}) + f(y_0) - f(x_{r_0}) < 0,$$

which contradicts (8). Thus we conclude that there exists $r' > 0$ such that $\|x_{r'}\| < r'$. Obviously, it is easy to see that for any $y \in K$ we can choose $\varepsilon \in (0, 1)$ such that $x_{r'} + \varepsilon(y - x_{r'}) \in K \cap B_{r'}$. Utilizing now (7), and conditions (ii), (iv), (v) in Theorem 2, we obtain for any $y \in K$ and some $s_{r'} \in Tx_{r'}$,

$$\begin{aligned}
0 &\leq \Psi(x_{r'} + \varepsilon(y - x_{r'}), x_{r'}; As_{r'}) + f(x_{r'} + \varepsilon(y - x_{r'})) - f(x_{r'}) \\
&\leq \varepsilon\Psi(y, x_{r'}; As_{r'}) + (1 - \varepsilon)\Psi(x_{r'}, x_{r'}; As_{r'}) + \varepsilon f(y) + (1 - \varepsilon)f(x_{r'}) - f(x_{r'}) \\
&= \varepsilon[\Psi(y, x_{r'}; As_{r'}) + f(y) - f(x_{r'})],
\end{aligned}$$

that is,

$$\Psi(y, x_{r'}; As_{r'}) + f(y) - f(x_{r'}) \geq 0, \quad \forall y \in K.$$

This completes the proof. □

Now, put $\Psi(x, y; z^*) = \langle z^*, \eta(x, y)\rangle$ for all $(x, y, z^*) \in K \times K \times E^*$ where $\eta : K \times K \to E$. Then the following corollaries follow immediately from the above Theorems 2 and 3, respectively.

Corollary 2. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E and let $T : K \to 2^{E^*}$ be a nonempty compact-valued multifunction such that for any $x, y \in K$,*

$$H(T(x + \lambda(y - x)), Tx) \to 0 \quad \text{as } \lambda \to 0^+,$$

where H is a Hausdorff metric defined on $CB(E^*)$. Suppose that the following conditions hold:

(i) *$A : E^* \to E^*$ is a continuous mapping;*
(ii) *$f : K \to R$ is a lower semicontinuous and convex functional;*
(iii) *$\eta(x, \cdot) : K \to E$ is continuous for each fixed $x \in K$;*
(iv) *$\eta(x, y) + \eta(y, x) = 0$ for each $(x, y) \in K \times K$;*
(v) *$\langle At, \eta(\cdot, y)\rangle : K \to R$ is a convex and lower semicontinuous function on K for each fixed $y \in K$ and $t \in Ty$;*
(vi) *T is generalized η–α-monotone with respect to A;*
(vii) *$\alpha(\cdot, y)$ is weakly lower semicontinuous for each fixed $y \in K$.*

Then problem (III) *has a solution.*

Corollary 3. *Let K be a nonempty closed convex subset of a real Banach space E and let $T : K \to 2^{E^*}$ be a nonempty compact-valued multifunction such that for any $x, y \in K$,*

$$H(T(x + \lambda(y - x)), Tx) \to 0 \quad \text{as } \lambda \to 0^+,$$

where H is a Hausdorff metric defined on $CB(E^*)$. Assume that conditions (i)–(vii) *of Corollary 1 are fulfilled together with:*

(viii) *$\Psi(x, y; z^*)(= \langle z^*, \eta(x, y)\rangle)$ is f-coercive with respect to T and A.*

Then problem (III) *has a solution.*

Acknowledgments

The authors thank the anonymous referees for careful reading and constructive suggestions which led to useful improvement of the work. The first author's research was partially supported by the National Science Foundation of China (10771141), Ph.D. Program Foundation of Ministry of Education of China (20070270004), and Science and Technology Commission of Shanghai Municipality grant (075105118). The second author's research was partially supported by a grant from the National Science Council.

References

[1] Q.H. Ansari, A.H. Siddiqi, J.C. YAO, Generalized vector variational-like inequalities and their scalarizations, in: F. Giannessi (Ed.), Vector Variational Inequalities and Vector Equilibria, 2002, pp. 17–37.
[2] Y.P. Fang, N.J. Huang, Variational-like inequalities with generalized monotone mappings in Banach spaces, J. Optim. Theory Appl. 118 (3) (2003) 327–337.
[3] V. Preda, M. Beldiman, A. Batatorescu, On variational-like inequalities with generalized monotone mappings, Preprint.
[4] I.V. Konnov, J.C. Yao, On the generalized vector variational inequality problems, J. Math. Anal. Appl. 206 (1997) 42–58.
[5] S. Schaible, Generalized monotonicity: Concepts and uses, in: F. Giannessi, A. Maugeri (Eds.), Variational Inequalities and Network Equilibrium Problems, Plenum Publishing Corporation, New York, 1995, pp. 289–299.
[6] I.V. Konnov, S. Schaible, Duality for equilibrium problems under generalized monotonicity, J. Optim. Theory Appl. 104 (2) (2000) 395–408.
[7] Q.H. Ansari, I.V. Konnov, J.C. Yao, Existence of a solution and variational principles for vector equilibrium problems, J. Optim. Theory Appl. 110 (3) (2001) 481–492.
[8] L.C. Zeng, S. Schaible, J.C. Yao, Iterative algorithm for generalized set-valued strongly nonlinear mixed variational-like inequalities, J. Optim. Theory Appl. 124 (2005) 725–738.
[9] J.C. Yao, Variational inequalities and generalized monotone operators, Math. Oper. Res. 19 (1994) 691–705.
[10] K. Fan, A generalization of Tychonoff's fixed-point theorem, Math. Ann. 142 (1961) 305–310.
[11] Y.Q. Chen, On the semimonotone operator theory and applications, J. Math. Anal. Appl. 231 (1999) 177–192.
[12] S.B. Nadler Jr., Multi-valued contraction mappings, Pacific J. Math. 30 (1969) 475–488.